Abstract

Let K be a compact subset of $\mathbb{C}^n$. The *polynomially convex hull* of K is defined as $\widehat{K} := \{z \in \mathbb{C}^n : |p(z)| \leq \sup_K |p| \quad \forall\ p \in \mathbb{C}[z_1, \ldots, z_n]\}$. The compact set K is said to be *polynomially convex* if $\widehat{K} = K$. A closed subset $E \subset \mathbb{C}^n$ is said to be *locally polynomially convex at* $z \in E$ if there exists a closed ball $\overline{\mathbb{B}}_z$ centred at z such that $E \cap \overline{\mathbb{B}}_z$ is polynomially convex. The aim of this thesis is to derive easily checkable conditions to detect polynomial convexity in certain classes of sets in $\mathbb{C}^n,\ n \geq 2$.

This thesis begins with the basic question: *Let S_1 and S_2 be two smooth, totally-real surfaces in $\mathbb{C}^2$ that contain the origin. If the union of their tangent planes is locally polynomially convex at the origin, then is $S_1 \cup S_2$ locally polynomially convex at the origin?* If $T_0S_1 \cap T_0S_2 = \{0\}$, then it is a folk result that the answer is, "Yes." We discuss an obstruction to the presumed proof, and use a different approach to provide a proof. When $dim_{\mathbb{R}}(T_0S_1 \cap T_0S_2) = 1$, it turns out that the positioning of the complexification of $T_0S_1 \cap T_0S_2$ controls the outcome in many situations. In general, however, local polynomial convexity of $S_1 \cup S_2$ also depends on the degeneracy of the contact of T_0S_j with S_j at 0, $j = 1, 2$. We establish a result showing this.

Next, we consider a generalization of Weinstock's theorem for more than two totally-real planes in $\mathbb{C}^2$. Using a characterization, recently found by Florentino, for simultaneous triangularizability over $\mathbb{R}$ of real matrices, we present a sufficient condition for local polynomial convexity at $0 \in \mathbb{C}^2$ of union of finitely many totally-real planes is $\mathbb{C}^2$.

The next result is motivated by an approximation theorem of Axler and Shields, which says that the uniform algebra on the closed unit disc $\overline{\mathbb{D}}$ generated by z and h — where h is a nowhere-holomorphic harmonic function on $\mathbb{D}$ that is continuous up to $\partial\mathbb{D}$ — equals $\mathcal{C}(\overline{\mathbb{D}})$. The abstract tools used by Axler and Shields make harmonicity of h an essential condition for their result. We use the concepts of plurisubharmonicity and polynomial convexity to show that, in fact, the same conclusion is reached if h is replaced by $h + R$, where R is a non-harmonic perturbation whose Laplacian is "small" in a certain sense. Ideas developed for the latter result, especially the role of plurisubharmonicity, lead us to our final result: a *characterization* for compact patches of smooth, totally-real graphs in $\mathbb{C}^{2n}$ to be polynomially convex.