Abstract

Let $X$ be the $n$ element set $\{1, 2, 3, \ldots, n\}$ and $\mathcal{F}$ a family of subsets of $X$. We call $\mathcal{F}$ a hypergraph on $X$. The set of all subsets of $X$ is denoted by $\mathcal{P}(X)$ and the set of $h$-element subsets of $X$ by $X^{(h)}$. When $\mathcal{F} \subseteq X^{(h)}$ we call $\mathcal{F}$ a $h$ uniform hypergraph. We also call the elements of $\mathcal{F}$ as edges. All hypergraphs mentioned here will be over $X$.

In 1948, de Bruijn and Erdos proved the following theorem [9]

**Theorem (N.G. de Bruijn and P. Erdos, 1948)**

If $\mathcal{F}$ is a hypergraph on $X$, such that $|E \cap F| = 1$ for all distinct pairs $E, F \in \mathcal{F}, E \neq F$, then $|\mathcal{F}| \leq n$.

This was the starting point of several investigations in Design Theory and Extremal Combinatorics. In Extremal Combinatorics, this led to the birth of the topic of intersection theorems for hypergraphs. A typical problem in this topic is to determine the maximum size of a hypergraph $\mathcal{F}$ with the property that $|E \cap F| \in S$, for every pair $E, F \in \mathcal{F}, E \neq F$, where $S$ is a given set of non-negative integers [3, 12]. It is clear that the de Bruijn - Erdos theorem determines the maximum size of a hypergraph $\mathcal{F}$ when $S = \{1\}$. Majumdar proved that the same bound holds even when $S = \{\lambda\}$ [56]. Even though these were the first published theorems in this topic, it was only after the publication of the celebrated theorem of Erdos, Ko and Rado (henceforth, the EKR theorem) that rapid advances took place in this topic. The EKR theorem determines the maximum size of a $h$-uniform hypergraph, with the intersection sizes of pairs of distinct edges of $\mathcal{F}$ belonging to $\{t, t+1, \ldots, h-1\}$. With his immutable style, Paul Erdos popularized this topic by raising several conjectures and prize problems. For a survey of results in this topic, consult the paper of Deza and Frankl or the lecture notes of Babai and Frankl [3, 12]. An important result in this topic is the following theorem of Frankl and Wilson, which was proved by them using linear algebraic arguments [32].
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Theorem (Frankl and Wilson, 1981)

If $\mathcal{F}$ is a hypergraph such that $|E \cap F| \in S$ for every $E, F \in \mathcal{F}$, $E \neq F$, where $S$ is a set of non-negative integers, then $|\mathcal{F}| \leq \sum_{i \leq |S|} \binom{n}{i}$

In the same year, Frankl and Furedi studied the problem of estimating the size of a hypergraph $\mathcal{F}$ such that $|E \cap F| \in \{1, 2, \ldots, k\}$ for every distinct pair of edges $E, F \in \mathcal{F}$ [23]. They proved that if $n \leq 2k + 2$ or $n > \frac{100k^2}{\log{(k+1)}}$ then $|\mathcal{F}| \leq \sum_{i=0}^{k} \binom{n-1}{i}$ When $k = 1$, the problem reduces to the de Bruijn - Erdos theorem, which states that the bound on the size of $\mathcal{F}$ is true for all $n$. This led them to make the following conjecture

Frankl-Furedi Conjecture

If $\mathcal{F}$ is a hypergraph on $X = \{1, 2, 3, \ldots, n\}$ such that $1 \leq |E \cap F| \leq k$ for all $E, F \in \mathcal{F}$, $E \neq F$, then $|\mathcal{F}| \leq \sum_{i=0}^{k} \binom{n-1}{i}$

Pyber verified the conjecture for $n \leq 2k + 2$ and for $6(k+1) \leq n \leq \frac{1}{8}(k+1)^2$ using a new type of permutation method [62]. His proof, even though elementary, seems to be involved.

In a recent note, Snevily verified the conjecture for $n \geq (k^4/2)(k+1)^2$ using partly, a linear algebraic method of Alon, Babai and Suzuki [69]. He also proved the conjecture for $k = 2$, assuming the verification of Frankl and Furedi for the case $n \leq 2k + 2$, for which Pyber had given a simple proof.

The main contribution of this thesis is to prove the Frankl-Furedi Conjecture. In order to prove the conjecture, we develop a new linear algebraic method.

The thesis is organized as follows. In Chapter 1, we give an introduction to some of the highlights in the topic of intersection theorems beginning with the de Bruijn - Erdos theorem and ending with some recent generalizations due to Alon, Babai and Suzuki [1]. We give an exposition of two important proof techniques in this area, one of which is due to Frankl and Wilson and the other, due to Alon, Babai and Suzuki [1, 32]. We also describe Snevily's verification of the Frankl-Furedi Conjecture in order to indicate the difficulty of the problem.

In Chapter 2, we give the proof of the Frankl-Furedi Conjecture. Our proof technique is easily described. We associate with each edge $E \in \mathcal{F}$ a real variable $x_E$. We derive an identity satisfied by the hypergraph $\mathcal{F}$ using its intersection properties. This hypergraph identity is simply a quadratic form involving squares of $x_E$'s and squares of real linear forms involving the $x_E$'s. Palisse derived a similar identity when the hypergraph $\mathcal{F}$ satisfies the property
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\[ |E \cap F| = \lambda \] for all distinct edges \( E, F \in \mathcal{F} \) using a simple counting argument [60]. From the hypergraph identity, we obtain a set of homogeneous linear equations. We then show that this defines the zero subspace of \( \mathbb{R}^{||\mathcal{F}||} \). Finally, the desired bound on \( |\mathcal{F}| \) is obtained from the bound on the number of homogeneous linear equations. Our method generalizes Palisz's method and can be viewed as a variant of the technique used by Tverberg to prove a result of Graham and Pollak [3].

In Chapter 3, we discuss our proof technique in detail. We highlight the key steps in our argument and lay the ground for proving some general intersection theorems for hypergraphs. We point out the limitations of our method and also give a useful proposition from the Calculus of Finite Differences. This can be used to prove binomial identities, which are an essential ingredient in the derivation of the hypergraph identity.

Chapter 4 contains some intersection theorems for hypergraphs. First, we modify our method to prove a theorem for a class of non-uniform hypergraphs satisfying certain properties, which depend on the size of the intersection of pairs of distinct edges. Next, we show that our proof technique can also be generalized to cover the case of uniform hypergraphs, by proving the uniform Ray-Chaudhuri - Wilson theorem. Finally, we prove a generalization of the uniform Ray-Chaudhuri - Wilson theorem. In order to prove this theorem, we derive a hypergraph identity in a different fashion.

In Chapter 5, we give the analogue of the uniform Ray-Chaudhuri - Wilson theorem and our generalization of it, to the class of full homogeneous lattices which have the Vandermonde property.

In Chapter 6, we list a few problems and some conjectures. In the light of our work, it would be interesting to prove other intersection theorems using our proof technique. We also state some general conjectures due to Frankl - Furedi, Pyber and Snevily.