Boxicity and Cubicity:  
A Study on Special Classes of Graphs

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Abstract

Let $\mathcal{F}$ be a family of sets. A graph $G$ is an intersection graph of sets from the family $\mathcal{F}$ if there exists a mapping $f : V(G) \to \mathcal{F}$ such that, $\forall u, v \in V(G)$, $(u, v) \in E(G) \iff f(u) \cap f(v) \neq \emptyset$. An interval graph is an intersection graph of a family of closed intervals on the real line. Interval graphs find application in diverse fields ranging from DNA analysis to VLSI design.

An interval on the real line can be generalized to a $k$ dimensional box or $k$-box. A $k$-box $B = (R_1, R_2, \ldots, R_k)$ is defined to be the Cartesian product $R_1 \times R_2 \times \cdots \times R_k$, where each $R_i$ is a closed interval on the real line. If each $R_i$ is a unit length interval, we call $B$ a $k$-cube. Thus, an interval is a 1-box and a unit length interval is a 1-cube. A graph $G$ has a $k$-box representation, if $G$ is an intersection graph of a family of $k$-boxes in $R^k$. Similarly, $G$ has a $k$-cube representation, if $G$ is an intersection graph of a family of $k$-cubes in $R^k$. The boxicity of $G$, denoted by box($G$), is the minimum positive integer $k$ such that $G$ has a $k$-box representation. Similarly, the cubicity of $G$, denoted by cub($G$), is the minimum positive integer $k$ such that $G$ has a $k$-cube representation. Thus, interval graphs are the graphs with boxicity equal to 1 and unit interval graphs are the graphs with cubicity equal to 1.

The concepts of boxicity and cubicity were introduced by F.S. Roberts in 1969. Deciding whether the boxicity (or cubicity) of a graph is at most $k$ is NP-complete even for a small positive integer $k$. Box representation of graphs finds application in niche overlap (competition) in ecology and to problems of fleet maintenance in operations research. Given a low dimensional box representation, some well known NP-hard problems become polynomial time solvable.

Attempts to find efficient box and cube representations for special classes of graphs can be seen in the literature. Scheinerman [6] showed that the boxicity of outerplanar graphs is at most 2. Thomassen [7] proved that the boxicity of planar graphs is bounded from above by 3. Cube representations of special classes of graphs like hypercubes and complete multipartite graphs were investigated in [5, 3, 4]. In this thesis, we present several bounds for boxicity and cubicity of special classes of graphs in terms of other graph parameters. The following are the main results shown in this work.

1. It was shown in [2] that, for a graph $G$ with maximum degree $\Delta$, $\text{cub}(G) \leq \lfloor 4(\Delta + 1) \log n \rfloor$. We show that, for a $k$-degenerate graph $G$, $\text{cub}(G) \leq (k + 2)\lceil \Delta \log n \rceil$. Since $k$ is at most $\Delta$ and can be much lower, this clearly is a stronger result. This bound is tight up to a constant factor.

2. For a $k$-degenerate graph $G$, we give an efficient deterministic algorithm that runs in $O(n^2 \log k)$ time to output an $O(k \log n)$ dimensional cube representation.

3. Crossing number of a graph $G$ is the minimum number of crossing pairs of edges, over all drawings of $G$ in the plane. We show that if crossing number of $G$ is $t$, then box($G$) is $O(t^{1/4} \log^{3/4} t)$. This bound is tight up to a factor of $O((\log t)^2)$.

4. We prove that almost all graphs have cubicity $O(d_{av} \log n)$, where $d_{av}$ denotes the average degree.

5. Boxicity of a $k$-leaf power is at most $k - 1$. For every $k$, there exist $k$-leaf powers whose boxicity is exactly $k - 1$. Since leaf powers are a subclass of strongly chordal graphs, this result implies that there exist strongly chordal graphs with arbitrarily high boxicity.

6. Otachi et al. [8] conjectured that chordal bipartite graphs (CBGs) have boxicity at most 2. We disprove this conjecture by exhibiting an infinite family of CBGs that have unbounded boxicity. We first prove that the bipartite power of a tree (which is a CBG) is a CBG and then show that there exist trees whose bipartite powers have high boxicity. Later in Chapter ??, we prove a more generic result in bipartite powering. We prove that, for every $k \geq 3$, the bipartite power of a bipartite, $k$-chordal graph is bipartite and $k$-chordal thus implying that CBGs are closed under bipartite powering.

7. Boxicity of a line graph with maximum degree $\Delta$ is $O(\Delta \log^2 \log^2 \Delta)$. This is a $\frac{\log^2 \Delta}{\log^2 \log \Delta}$ factor improvement over the best known upper bound for boxicity of any graph [1]. We also prove a non-trivial lower bound for the boxicity of a $d$-dimensional hypercube.